Math 564: Real analysis and measure theory
Lecture 15

Def. Let $(X, \mathcal{B}, \mu)$ be a measure space, where $\mathcal{B} := \mathrm{Meas}_\mu$, i.e. μ is complete.
A μ-measurable function is said to be *μ-integrable* if $\int |f| \, d\mu < \infty$. In this case,
$$\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu. \quad (\text{Note that } |f| = f_+ + f_-.)$$

Note. $|f| = f_+ + f_-$ so $\int |f| \, d\mu = \int f_+ \, d\mu + \int f_- \, d\mu$. In particular, $\left| \int f \, d\mu \right| \le \int |f| \, d\mu$.

Remark. If f is a complex-valued function, then $\int f \, d\mu := \int \mathrm{Re} f \, d\mu + i \int \mathrm{Im} f \, d\mu$.

We denote the space of μ-integrable μ-measurable functions by $L^1(X, \mu) := L^1(X, \mathcal{B}, \mu)$.
Commonly $L^1(X, \mu)$ denotes the quotient of the set of all integrable functions by the equivalence relation $f = g$ a.e. We will typically use $L^1(X, \mu)$ as literally the space of all μ-integrable functions, with the understanding that we could have used the quotient.

Define a (pseudo) norm on $L^1(X, \mu)$ by $\|f\|_1 := \int |f| \, d\mu$. Call this the L^1-norm.

Observation. $L^1(X, \mu)$ equipped with $\|\cdot\|_1$ is a (pseudo) normed vector space, i.e. for all $f, g \in L^1(X, \mu)$, we have:

(i) $\|f\|_1 \ge 0$ and $\|f\|_1 = 0 \iff f = 0$ a.e.

(ii) $\|\alpha f\|_1 = |\alpha| \, \|f\|_1$ for all $\alpha \in \mathbb{R}$.

(iii) Triangle inequality: $\|f + g\|_1 \le \|f\|_1 + \|g\|_1$.

Proof. For (iii), note $\|f + g\|_1 = \int |f + g| \, d\mu \overset{\text{mon. of integ.}}{\le} \int |f| + |g| \, d\mu \overset{\text{linearity}}{=} \int |f| \, d\mu + \int |g| \, d\mu = \|f\|_1 + \|g\|_1$. $\square$

As always, a (pseudo) normed vector space is also a (pseudo) metric space by
$$d_1(f, g) := \|f - g\|_1.$$

Thus, we say that a sequence $(f_n) \subseteq \mathcal{L}^1(X,\mu)$ converges in $\mathcal{L}^1$-norm to $f \in \mathcal{L}^1(X,\mu)$ if it converges in the (pseudo) metric d_1, i.e. $f_n \to_{\mathcal{L}^1} f$ if $d_1(f,f_n) := \|f - f_n\|_1 \to 0$ as $n \to \infty$.

<u>Observation.</u> $f_n \to_{\mathcal{L}^1} f$ implies (in particular) $\int f_n \, d\mu \to \int f \, d\mu$.

Proof. $\left| \int f \, d\mu - \int f_n \, d\mu \right| = \left| \int (f - f_n) \, d\mu \right| \leq \int |f - f_n| \, d\mu = \|f - f_n\|_1 \to 0.$ $\square$

<u>Notation.</u> When μ is the counting measure on X, then $\ell^1(X) := \mathcal{L}^1(X, \text{counting measure})$.

<u>Examples.</u> (a) $\ell^1(\mathbb{N}) :=$ absolutely summable sequences. Indeed, for $f \in \ell^1(\mathbb{N})$ and μ counting measure on $\mathbb{N}$,

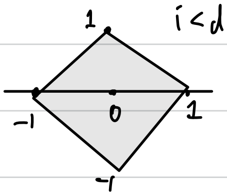
$$\int f \, d\mu = \sum_{n \in \mathbb{N}} f(n) \qquad \text{(HW)}$$

so $\|f\|_1 := \sum_{n \in \mathbb{N}} |f(n)|$.

(b) Let $d \in \mathbb{N}$, so $d = \{0, 1, \dots, d-1\}$. What is $\ell^1(d)$? It's just $\mathbb{R}^d$ with the 1-norm:

$$\|f\|_1 := \sum_{i<d} |f(i)|.$$

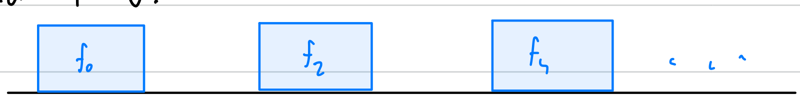
The unit ball is



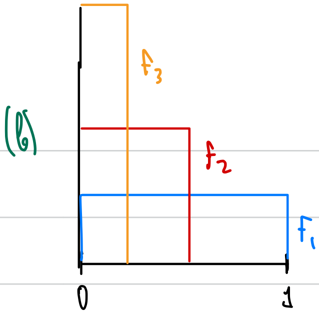
<u>Pointwise v.s. $\mathcal{L}^1$ convergence.</u>

<u>Examples of disagreement.</u> All examples below are in $\mathcal{L}^1(\mathbb{R}, \lambda)$.

(a) Let $f_n := \mathbb{1}_{[n,n+1)}$ and $f \equiv 0$.



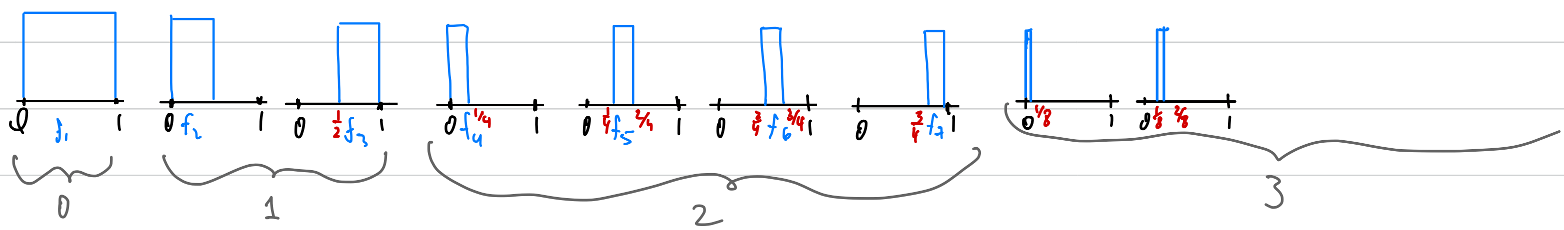
$f_n \to 0$ pointwise but $\int f_n \, d\lambda = 1$ for all $n \in \mathbb{N}$ while $\int f \, d\mu = 0$. Also $\|f_n - f_m\|_1 = 1$ for $n \neq m$.



(b) Let $f_n := n \cdot \mathbb{1}_{[0,\frac{1}{n}]}$ for $n \in \mathbb{N}^+$. Then $f_n \to 0$ pointwise but $\int f_n d\mu = \frac{1}{n} \cdot n = 1$. Hence $f_n \not\to_{L^1} 0$.

(c) Here is a example of $f_n \to_{L^1} 0$ but not at any point.

$f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,1/2]}$, $f_6 = \chi_{[1/2,3/4]}$, $f_7 = \chi_{[3/4,1]}$, and in general, $f_n = \chi_{[j/2^k,\,(j+1)/2^k]}$ where $n = 2^k + j$ with $0 \le j < 2^k$.



$\|f_n - 0\|_1 = \int f_n d\lambda = \frac{1}{2^k}$ if f_n belongs to the k^{th} group, so $f_n \to_{L^1} 0$. But $(f_n(x))$ diverges for each $x \in [0,1]$ because $(f_n(x))$ has ∞ many 0 and 1.

We will discuss example (c) and its fix later, after introducing convergence in measure, but we can fix examples (a)-(b) now. Note that in (a)-(b), there is no integrable $g \ge 0$ dominating all of the $|f_n|$. Indeed in (a) such a g has to be $\ge \mathbb{1}_{[0,\infty)}$, and in (b), $\ge \max\{f_n\} \sim \frac{1}{x}$.

Dominated Convergence Theorem (DCT). Let f_n, f be μ-measurable functions with $f_n \to f$ a.e. If there is dominating $g \in L^1(X,\mu)$, i.e. $g \ge 0$ and $|f_n| \le g$ for all $n \in \mathbb{N}$, then $f_n, f \in L^1(X,\mu)$ and

$$\int f_n d\mu \to \int f d\mu \quad \text{as} \quad n \to \infty. \qquad (*)$$

In fact, $f_n \to_{L^1} f$ as $n \to \infty$.

Proof. The condition $|f_n| \le g$ implies $|f| \le g$ a.e. hence $f_n, f \in L^1$.
Fatou applied to $|f_n|$ gives $\int |f| d\mu \le \liminf_n \int |f_n| d\mu$. Other non-negative sequences are $g + f_n$ and $g - f_n$, and Fatou applied to each gives:

$$\cancel{\int g\,d\mu} + \int f\,d\mu = \int g+f\,d\mu \leq \liminf_n \int (g+f_n)\,d\mu = \cancel{\int g\,d\mu} + \liminf_n \int f_n\,d\mu.$$

$$\cancel{\int g\,d\mu} - \int f\,d\mu = \int g-f\,d\mu \leq \liminf_n \int g-f_n\,d\mu = \int g\,d\mu + \liminf_n -\int f_n\,d\mu = \cancel{\int g\,d\mu} - \limsup_n \int f_n\,d\mu.$$

So $\limsup_n \int f_n\,d\mu \leq \int f\,d\mu \leq \liminf_n \int f_n\,d\mu$, so $\lim_n \int f_n\,d\mu = \int f\,d\mu$.

To get the L^1 convergence, apply (*) to $|f-f_n|$. Indeed, $|f-f_n| \leq |f|+|f_n| \leq 2g$
Hence by (*),

$$\|f-f_n\|_1 = \int |f-f_n|\,d\mu \to \int \lim_n |f-f_n|\,d\mu = \int 0\,d\mu = 0, \text{ so } f_n \to_{L^1} f. \quad \square$$

L^1 as a (pseudo) metric space.

We analyze dense families in L^1 as well as whether the L^1 metric is complete.

Prop. Simple functions are dense in L^1 (in the L^1 metric).

Proof. If $f \in L^1(X,\mu)$ then $\exists$ sequence (s_n) of simple functions $s_n \to f$ pointwise and $|s_n| \leq |f|$, hence by DCT, $s_n \to_{L^1} f$ as $n \to \infty$. $\square$

Def. A measure space $(X, \mathcal{B}, \mu)$ is said to be *countably generated* if Meas_μ is separable as a pseudo metric space wrt $d_\mu(A,B) := \mu(A \Delta B)$, i.e. there exists a ctbl $\mathcal{G} \subseteq \mathrm{Meas}_\mu$ such that for each $M \in \mathrm{Meas}_\mu$ and $\varepsilon > 0$ $\exists$ $A \in \mathcal{G}$ with $d_\mu(M,A) < \varepsilon$.

Prop. For a σ-finite measure space $(X, \mathcal{B}, \mu)$, if $\mathcal{B}$ is ctbly generated as a σ-algebra then $(X, \mathcal{B}, \mu)$ is countably generated.

Proof. Follows from the uniqueness part of Carathéodory's theorem. Details done on the midterm. $\square$

Caution. The converse of the last proposition isn't true: there are ctbly generated $(X,\mathcal{B},\mu)$ s.t. $\mathcal{B}$ isn't ctbly generated as a σ-algebra.